A supplementary material for the paper "Depolarization field of spheroidal particles"

Alexander Moroz*
Wave-scattering.com

- the dot on the line just below Eq. (11) of [1] should be replaced by comma - the sentence continues further.
- ℓ_E in Eqs. (12), (35) of [1] is generically the spheroid dimension along the axis along which the external field $\mathbf{E}_0$ has been applied.
- the 2nd line above Eq. (55) on p. 524 of [1] should refer to Eq. (37) and not to Eq. (55) of [1].

1 Summary of elementary formulas

In the case of D_z, the integration over the ρ coordinate can be performed on using the following identities:

$$\int \frac{x\,dx}{(a^2+x^2)^{n+1/2}} = -\frac{1}{(2n-1)}\frac{1}{(a^2+x^2)^{n-1/2}}, \quad (1)$$

$$\int \frac{x^3\,dx}{(a^2+x^2)^{3/2}} = \frac{x^2+2a^2}{\sqrt{a^2+x^2}}, \quad (2)$$

which imply

$$\int \frac{2a^2+x^2}{(a^2+x^2)^{3/2}}\,x\,dx = \frac{x^2}{\sqrt{a^2+x^2}}. \quad (3)$$

The respective integrals (1) and (2) are given as (1.2.43.12) and (1.2.43.20) by [2].

All the integrals here and below can be easily verified by differentiation. In arriving at final results, one makes repeated use of the following formula

$$\sqrt{1-e^2} = \begin{cases} \frac{a}{c}, & \text{prolate} \\ \frac{c}{a}, & \text{oblate} \end{cases} \quad (4)$$

*http://www.wave-scattering.com

1.1 Quadrature formulas for a prolate spheroid

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right|, \tag{5}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2+x^2}} = \frac{x\sqrt{a^2+x^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|. \tag{6}$$

The respective integrals (5) and (6) are given as (1.2.43.13) and (1.2.43.15) by [2].

1.2 Quadrature formulas for an oblate spheroid

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{|a|}, \tag{7}$$

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{|a|}. \tag{8}$$

The respective integrals (7) and (8) are given as (1.2.48.12) and (1.2.48.14) by [2].

1.3 Quadrature formulas for the sum rule

In the case of the sum on the l.h.s. of the sum rule in Eq. (37), because the sum of directional cosines is

$$\sum_{j=1}^{3}\cos^2\theta_j = 1, \tag{9}$$

the task amounts to performing the integral

$$\begin{aligned}(c/a)(D_x+D_y)+D_z &= 2\int\frac{dV}{r}\\ &= 8\pi ac\int_0^1 dz'\int_0^{\sqrt{1-(z')^2}}\frac{\rho\,d\rho}{[\rho^2+(c^2/a^2)(z')^2]^{1/2}}.\end{aligned} \tag{10}$$

As usual, in the last integral the trivial integration over φ has already been performed and the use was made of the mirror symmetry $z'\to -z'$ of the respective integrands. The ρ-integration can be performed on using the elementary identity:

$$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}. \tag{11}$$

The remaining z'-integration can be performed on using the elementary identities

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|, \tag{12}$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{|a|}. \tag{13}$$

The respective integrals (12), (13) are for *prolate* and *oblate* spheroids, and are given as (1.2.41.8) and (1.2.46.8) by [2].

In the case of $D_{z;rn}$ given by Eq. (55) of [1] the corresponding $D_{\perp;rn}$ *cannot* be determined by the sum rule in Eq. (37) of [1]. The latter presumes that the term proportional to k^2 has the form as in Eq. (15) of [1]. In the case of the k^2-dependent term given in Eq. (53) of [1] one is required to perform volume integrals of $\cos^4\theta/r$. Because no simple relation exists for the sum of directional cosines

$$\sum_{j=1}^{3}\cos^4\theta_j, \tag{14}$$

the corresponding volume integral could not have been performed.

1.4 $\cos^{2n}\theta$ quadrature formulas

For $\mathbf{E}_0$ polarized along the z-axis of rotational symmetry one finds with $\cos\theta = z/r$ in the case of a *prolate* spheroid

$$\int \frac{\cos^{2n}\theta}{2r}\,dV = \pi a^2 \begin{cases} \frac{1}{e}\,\mathrm{arctanh}\; e, & n=0 \\ \frac{1}{1-e^2}\,L_z, & n=1 \\ \frac{1}{3e^2}\left[\frac{3L_z}{1-e^2}-1\right], & n=2 \end{cases} \tag{15}$$

whereas, for an *oblate* spheroid,

$$\int \frac{\cos^{2n}\theta}{2r}\,dV = \pi a^2 \begin{cases} \frac{\sqrt{1-e^2}}{e}\,\arcsin e, & n=0 \\ (1-e^2)L_z, & n=1 \\ \frac{1-e^2}{3e^2}\left[1-3(1-e^2)L_z\right], & n=2 \end{cases} \tag{16}$$

For a sphere the above formulas reduce to

$$\int \frac{\cos^{2n}\theta}{2r}\,dV = \frac{\pi a^2}{2n+1}\cdot \tag{17}$$

1.5 Asymptotic expansions

$$\begin{aligned} \mathrm{arctanh}\; z &= \frac{1}{2}\left[\ln(1+z)-\ln(1-z)\right] \\ &= \sum_{l=0}^{\infty}\frac{z^{2l+1}}{2l+1} \sim z+\frac{z^3}{3}+\frac{z^5}{5}+\mathcal{O}(z^7), \end{aligned} \tag{18}$$

$$\arcsin z \sim z+\frac{z^3}{6}+\frac{3}{40}z^5+\mathcal{O}(z^7), \tag{19}$$

$$\sqrt{1-z^2} \sim 1-\frac{z^2}{2}-\frac{z^4}{8}+\mathcal{O}(z^6). \tag{20}$$

References

[1] A. Moroz, "Depolarization field of spheroidal particles," J. Opt. Soc. Am. B **26**, 517-527 (2009).

[2] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, 2nd ed (Gordon and Breach, London, 1988).