On-line supplementary material for the article "Electron mean-free path in metal coated nanowires"

Alexander Moroz
Wave-scattering.com

1. Notation and definitions

One has [cf Eqs. (17.3.11), (17.3.12), (17.3.26), (17.3.27) of [2]]

$$K(q^2) \sim \frac{\pi}{2}\left(1+\frac{q^2}{4}\right) \qquad (q\to 0), \tag{1}$$

$$E(q^2) \sim \frac{\pi}{2}\left(1-\frac{q^2}{4}\right) \qquad (q\to 0), \tag{2}$$

$$K(q^2)-E(q^2) \sim \frac{\pi}{4}q^2 \qquad (q\to 0), \tag{3}$$

$$K(q^2) \sim \frac{1}{2}\ln\frac{16}{1-q^2} = -\frac{1}{2}\ln(1-q^2)+2\ln 2 \qquad (q\to 1_-), \tag{4}$$

$$E(q^2) \sim 1-\frac{(1-q^2)}{4}\ln(1-q^2)-(1-q^2)\left(\frac{1}{4}-\ln 2\right) \qquad (q\to 1_-). \tag{5}$$

$$\operatorname{arcsinh} z = \ln(z+\sqrt{z^2+1}). \tag{6}$$

For $|x|\le 1$,

$$\operatorname{arcsinh} ix = i\arcsin x. \tag{7}$$

2. Recasting up Wolfram integrator formulae

A hint at arriving at the expressions (55) and (55) of [1] for the integrals C_1 and C_2 has been provided by Wolfram integrator [3].

2.A. Integral Eq. (55) of [1]

Given the input: `(cos x*Sqrt[b-cos x])/(cos x-q)`, the Wolfram integrator [3] generates a formula that can be simplified as follows. Let

$$\begin{aligned} X &= 1 - \frac{\beta - \cos\theta}{\beta - 1}, \\ Y &= 1 - \frac{\beta - \cos\theta}{\beta - q}. \end{aligned} \tag{8}$$

Then

$$\begin{aligned} &(1+q-\beta)Y - q + (\beta-1)YX = \\ &\quad 1 + q - \beta - q + \beta - 1 + \frac{(\beta-\cos\theta)}{(\beta-1)(\beta-q)}\{ \\ &\quad -(1+q-\beta)(\beta-1) - (\beta-1)(\beta-q+\beta-1) + \\ &\quad (\beta-1)(\beta-\cos\theta)\} \\ &= \frac{(\beta-\cos\theta)}{(\beta-q)}[-(1+q-\beta+\beta-q+\beta-1) + (\beta-\cos\theta)] \\ &= \frac{(\beta-\cos\theta)}{(\beta-q)}(-\beta+\beta-\cos\theta) = -\frac{(\beta-\cos\theta)}{(\beta-q)}\cos\theta, \end{aligned} \tag{9}$$

$$\begin{aligned} \sqrt{X} &= i\sqrt{\frac{1-\cos\theta}{\beta-1}}, \\ Y &= \frac{\cos\theta - q}{\beta - q}, \end{aligned} \tag{10}$$

$$\frac{d}{d\theta}\arcsin\sqrt{\frac{\beta-\cos\theta}{\beta+1}} = \frac{1}{2}\frac{\sin\theta}{\sqrt{1+\cos\theta}\sqrt{\beta-\cos\theta}}. \tag{11}$$

Eventually,

$$\frac{(1+q-\beta)Y - q + (\beta-1)YX}{Y\sqrt{X}}\left(\frac{d}{d\theta}\arcsin\sqrt{\frac{\beta-\cos\theta}{b+1}}\right)$$

$$
\begin{aligned}
&= \frac{i}{2}\sqrt{\beta-1}\frac{\sqrt{\beta-\cos\theta}}{\cos\theta-q}\,\cos\theta\frac{\sin\theta}{\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}} \\
&= \frac{i}{2}\sqrt{\beta-1}\frac{\sqrt{\beta-\cos\theta}}{\cos\theta-q}\,\cos\theta. \qquad (12)
\end{aligned}
$$

2.B. Integral Eq. (57) of [1]

After recasting and rearranging the terms in the expression supplied by the Wolfram integrator [3] for a given input: `(Sqrt[1- x^2])/((x-q)Sqrt[b- x])`, the integral on the r.h.s. of defining Eq. (54) of [1] for C_2 can be expressed as

$$
\begin{aligned}
&\int \frac{\sqrt{1-x^2}}{(x-q)(\beta-x)^{1/2}}\,dx = 2i\,\frac{(\beta-x)^{3/2}}{\sqrt{\beta+1}\sqrt{1-x^2}(\beta-q)} \\
&\quad\left\{\frac{i\sqrt{\beta+1}}{(\beta-x)^2}[\beta^3-\beta^2 q-2\beta^2(\beta-x)+2\beta q(\beta-x)+q-q(\beta-x)^2-\beta+\beta(\beta-x)^2]\right. \\
&\quad+\frac{(1-q^2)\sqrt{1-x^2}}{(\beta-x)^{3/2}}\,\Pi\left(\frac{\beta-q}{\beta+1};\Phi_2\middle|\frac{\beta-1}{\beta+1}\right) \\
&\quad-\frac{(1-q)(\beta+1)\sqrt{1-x^2}}{(\beta-x)^{3/2}}\,F\left(\Phi_2\middle|\frac{\beta-1}{\beta+1}\right) \\
&\quad\left.+\,\frac{(\beta-q)(\beta+1)\sqrt{1-x^2}}{(\beta-x)^{3/2}}\,E\left(\Phi_2\middle|\frac{\beta-1}{\beta+1}\right)\right\}, \qquad (13)
\end{aligned}
$$

where $\Phi_2=\Phi_2(x)$ here is given by Eq. (58) of [1]. Now

$$
\begin{aligned}
&\beta^3-\beta^2 q-2\beta^2(\beta-x)+2\beta q(\beta-x)+q-q(\beta-x)^2-\beta+\beta(\beta-x)^2 \\
&= -(\beta-q)(1-x^2). \qquad (14)
\end{aligned}
$$

After rearranging remaining terms one then arrives at Eq. (57) of [1].

References

1. A. Moroz, "Electron mean-free path in metal coated nanowires," J. Opt. Soc. Am. B **28**(5), 1130-1138 (2011).
2. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1973).
3. Wolfram Mathematica online integrator available at http://integrals.wolfram.com